

INTEGRABILITY OF THE FOURIER TRANSFORM: FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. Certain relations between the Fourier transform of a function of bounded variation and the Hilbert transform of its derivative are revealed. The widest subspaces of the space of functions of bounded variation are indicated in which the cosine and sine Fourier transforms are integrable.

1. INTRODUCTION

We are going to compare the Fourier transform of a function of bounded variation and the Hilbert transform of a related function. For this, let us start with some known results. The first one is given in [7, Thm.2] (see also [2]). We define the following T -transform of a function $g : \mathbb{R} = [0, \infty) \rightarrow \mathbb{C}$:

$$Tg(t) = \int_0^{t/2} \frac{g(t+s) - g(t-s)}{s} ds,$$

where the integral is understood in the improper (principal value) sense, that is, as $\lim_{\delta \rightarrow 0+} \int_\delta$.

Theorem 1. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be locally absolutely continuous, of bounded variation and $\lim_{t \rightarrow \infty} f(t) = 0$. Let also $Tf' \in L^1(\mathbb{R}_+)$. Then the cosine Fourier transform of f*

$$(1) \quad \widehat{f}_c(x) = \int_0^\infty f(t) \cos xt \, dt$$

is Lebesgue integrable on \mathbb{R}_+ , with

$$(2) \quad \|\widehat{f}_c\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{L^1(\mathbb{R}_+)} + \|Tf'\|_{L^1(\mathbb{R}_+)},$$

and for the sine Fourier transform, we have, with $x > 0$,

$$(3) \quad \widehat{f}_s(x) = \int_0^\infty f(t) \sin xt \, dt = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + F(x),$$

where

$$(4) \quad \|F\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{L^1(\mathbb{R}_+)} + \|Tf'\|_{L^1(\mathbb{R}_+)}.$$

Here and in what follows we use the notation “ \lesssim ” and “ \gtrsim ” as abbreviations for “ $\leq C$ ” and “ $\geq C$ ”, with C being an absolute positive constant.

Let us now turn to the Hilbert transform of an integrable function g

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$$(5) \quad \mathcal{H}g(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{t-x} dt,$$

where the integral is also understood in the improper (principal value) sense, now as $\lim_{\delta \rightarrow 0+} \int_{|t-x|>\delta}$. It is not necessarily integrable, and when it is, we say that g is in the (real) Hardy space $H^1(\mathbb{R})$. If $g \in H^1(\mathbb{R})$, then

$$(6) \quad \int_{\mathbb{R}} g(t) dt = 0.$$

It was apparently first mentioned in [6].

An odd function always satisfies (6). However, not every odd integrable function belongs to $H^1(\mathbb{R})$, for a counterexample see, e.g., [8]. When in the definition of the Hilbert transform (5) the function g is odd, we will denote this transform by \mathcal{H}_0 , and it is equal to

$$(7) \quad \mathcal{H}_0 g(x) = \frac{2}{\pi} \int_0^\infty \frac{tg(t)}{t^2 - x^2} dt.$$

If it is integrable, we will denote the corresponding Hardy space by $H_0^1(\mathbb{R})$.

Since

$$\mathcal{H}_0 g(x) = Tg(x) + \Gamma(x),$$

where Γ is such that

$$\int_0^\infty |\Gamma(x)| dx \lesssim \int_0^\infty |g(t)| dt,$$

the right-hand sides of (2) and (4) can be treated as $\|f'\|_{H_0^1(\mathbb{R}_+)}$. This has been observed in [7] and later on in [2].

The space of integrable functions g with integrable Tg , or just $H_0^1(\mathbb{R}_+)$, is one of the widest spaces the belonging of the derivative f' to which ensures the integrability of the cosine Fourier transform of f . However, the possibility of existence (or non-existence) of a wider space of such type is of considerable interest. Let us show that such a space does exist, moreover, it is the widest possible, at least provides a necessary and sufficient condition for the integrability of the cosine Fourier transform. In fact, it has in essence been introduced (for different purposes) in [4] as

$$(8) \quad Q = \{g : g \in L^1(\mathbb{R}), \int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|} dx < \infty\}.$$

With the obvious norm

$$\|g\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|} dx$$

it is a Banach space and ideal in $L^1(\mathbb{R})$. What we will actually use is the space Q_0 of the odd functions from Q

$$(9) \quad Q_0 = \{g : g \in L^1(\mathbb{R}), g(-t) = -g(t), \int_0^\infty \frac{|\widehat{g}_s(x)|}{x} dx < \infty\};$$

such functions naturally satisfy (6).

Theorem 2. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be locally absolutely continuous, of bounded variation and $\lim_{t \rightarrow \infty} f(t) = 0$. Then the cosine Fourier transform of f given by (1) is Lebesgue integrable on \mathbb{R}_+ if and only if $f' \in Q_0$.*

The situation is more delicate with the sine Fourier transform, where a sort of asymptotic relation can be obtained. In what follows we shall denote

$$(10) \quad \mathcal{T}_g(x) = \frac{\widehat{g}_s(x)}{x}.$$

Theorem 3. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be locally absolutely continuous, of bounded variation and $\lim_{t \rightarrow \infty} f(t) = 0$. Then for the sine Fourier transform of f given in (3) there holds for any $x > 0$*

$$(11) \quad \widehat{f}_s(x) = \int_0^\infty f(t) \sin xt \, dt = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + \mathcal{H}_0 \mathcal{T}_{f'}(x) + G(x),$$

where

$$(12) \quad \|G\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{L^1(\mathbb{R}_+)}.$$

This theorem makes it natural to consider a Hardy type space $H_Q^1(\mathbb{R}_+)$ which consists of Q_0 functions g with integrable $\mathcal{H}_0 \mathcal{T}_g$.

Corollary 4. *Let a function f satisfy the assumptions of Theorem 3 and such that $f' \in H_Q^1(\mathbb{R}_+)$. Then*

$$(13) \quad \widehat{f}_s(x) = \frac{1}{x} f\left(\frac{\pi}{2x}\right) + G(x),$$

where

$$(14) \quad \|G\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{H_Q^1(\mathbb{R}_+)}.$$

Technically, this is an obvious corollary of Theorem 3. We shall discuss it in Section 3.

2. PROOFS

Proof of Theorem 2. The assumptions of the theorem give a possibility to integrate by parts. This yields

$$\widehat{f}_c(x) = -\frac{1}{x} \int_0^\infty f'(t) \sin xt \, dt = -\frac{1}{x} \widehat{f}'_s(x).$$

Integrating both sides over \mathbb{R}_+ completes the proof. □

Proof of Theorem 3. Let us start with integration by parts in

$$\begin{aligned} \int_0^{\frac{\pi}{2x}} f(t) \sin xt \, dt &= \frac{1 - \cos xt}{x} f(t) \Big|_0^{\frac{\pi}{2x}} + \frac{1}{x} \int_0^{\frac{\pi}{2x}} f'(t) [\cos xt - 1] \, dt \\ &= \frac{1}{x} f\left(\frac{\pi}{2x}\right) + \frac{1}{x} \int_0^{\frac{\pi}{2x}} f'(t) [\cos xt - 1] \, dt. \end{aligned}$$

The last value is bounded by $\int_0^{\frac{\pi}{2x}} t |f'(t)| \, dt$, and

$$(15) \quad \int_0^\infty \int_0^{\frac{\pi}{2x}} t |f'(t)| \, dt \, dx = \frac{\pi}{2} \int_0^\infty |f'(t)| \, dt.$$

Let us now consider

$$(16) \quad I = I(x) = \int_{\frac{\pi}{2x}}^\infty f(t) \sin xt \, dt.$$

We will start with the following statement.

Lemma 1. *There holds*

$$\mathcal{H}_0 \mathcal{T}_{f'}(x) = \frac{2}{x\pi} \int_0^\infty f'(t) \sin xt \int_{xt}^\infty \frac{\cos v}{v} \, dv \, dt + \frac{2}{x\pi} \int_0^\infty f'(t) \cos xt \int_0^{xt} \frac{\sin v}{v} \, dv \, dt.$$

Proof of Lemma 1. We have

$$(17) \quad \mathcal{H}_0 \mathcal{T}_{f'}(x) = \frac{2}{\pi} \int_0^\infty \widehat{f}'_s(u) \frac{1}{u^2 - x^2} \, du$$

$$(18) \quad = \frac{2}{\pi} \int_0^\infty f'(t) \int_0^\infty \frac{1}{u^2 - x^2} \sin ut \, du \, dt.$$

Denoting, as usual,

$$\text{Ci}(u) = - \int_u^\infty \frac{\cos t}{t} \, dt$$

and

$$\text{Si}(u) = \int_0^u \frac{\sin t}{t} \, dt = \frac{\pi}{2} - \int_u^\infty \frac{\sin t}{t} \, dt,$$

we will make use of the formula (see [1, Ch.II, (18)])

$$\int_0^\infty \frac{1}{a^2 - x^2} \sin yx \, dx = \frac{1}{a} [\sin ay \text{Ci}(ay) - \cos ay \text{Si}(ay)],$$

where the integrals is understood in the principal value sense and $a, y > 0$. We apply this formula to the inner integral on the right-hand side of (17), with $a = x$ and $y = t$. Using the expressions for Ci and Si, we complete the proof of the lemma. \square

With this in hand, we are going to prove that

$$(19) \quad I(x) = \mathcal{H}_0 \mathcal{T}_{f'}(x) + G(x),$$

where

$$(20) \quad \|G\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{L^1(\mathbb{R}_+)}.$$

We denote the two summands in the expression obtained in the lemma by I_1 and I_2 . For both, we make use of the fact that

$$\int_{xt}^{\infty} \frac{\cos v}{v} dv = O\left(\frac{1}{xt}\right).$$

The same true when $\cos v$ is replaced by $\sin v$. When $t \geq \frac{1}{x}$ we have

$$(21) \quad \int_0^{\infty} \frac{1}{x} \int_{\frac{1}{x}}^{\infty} |f'(t)| \frac{1}{xt} dt dx = \int_0^{\infty} |f'(t)| t \int_{\frac{1}{t}}^{\infty} \frac{1}{x^2} dx = \int_0^{\infty} |f'(t)| dt.$$

When $t \leq \frac{1}{x}$ we split the inner integral into two. First,

$$\int_1^{\infty} \frac{\cos v}{v} dv = O(1),$$

and using $\left| \frac{\sin xt}{x} \right| \leq t$, we arrive at the estimate similar to (15). Further, we have

$$\int_{xt}^1 \left| \frac{\cos v}{v} \right| dv = O\left(\ln \frac{1}{xt}\right).$$

By this, integrating in x over $(0, \infty)$, we have to estimate

$$\int_0^{\infty} |f'(t)| t \int_0^{1/t} \ln \frac{1}{xt} dx dt = \int_0^{\infty} |f'(t)| dt.$$

Here we use that

$$\int_0^{1/t} \ln \frac{1}{xt} dx = \frac{1}{t}.$$

In conclusion, I_1 can be treated as G .

Let us proceed to I_2 . Using that

$$\frac{1}{x} \left| \int_0^{xt} \frac{\sin v}{v} dv \right| = O(t),$$

we arrive for $t \leq \frac{\pi}{2x}$ at (15). Let now $t \leq \frac{\pi}{2x}$. We have

$$\int_0^{xt} \frac{\sin v}{v} dv = \frac{\pi}{2} - \int_{xt}^{\infty} \frac{\sin v}{v} dv.$$

Now

$$\frac{2}{x\pi} \int_0^\infty f'(t) \cos xt \, dt \frac{\pi}{2} = I.$$

For the integral $\int_{xt}^\infty \frac{\sin v}{v} \, dv$, the estimates are exactly like those in (21).

Combining (19) and estimates before Lemma 1, we complete the proof of the theorem. \square

3. DISCUSSION

Discussion and comments are in order. At first sight, Theorem 2 does not seem to be a result at all, at most a technical reformulation of (1). This could be so but not after the appearance of the analysis of Q in [4]. Indeed, the well-known extension of Hardy's inequality (see, e.g., [3, (7.24)])

$$(22) \quad \int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|} \, dx \lesssim \|g\|_{H^1(\mathbb{R})}$$

implies

$$(23) \quad H^1(\mathbb{R}) \subseteq Q \subseteq L_0^1(\mathbb{R}),$$

where the latter is the subspace of g in $L^1(\mathbb{R})$ which satisfy the cancelation property (6).

It is worth noting that (23) immediately proves (2) from Theorem 1. The initial proof in [7] is essentially more complicated.

It is doubtful that Q (or Q_0) may be defined in terms of f itself rather than its Fourier transform, therefore it is of interest to find certain proper subspaces of Q_0 wider than H^1 belonging to which is easily verifiable. We mention the paper [9] in which a family of subspaces between H^1 and L^1 is introduced and duality properties of that family are studied. However, it is not clear how to compare that family with Q_0 .

Back to Theorem 3, let us analyze (19). On the one hand, we have

$$\int_0^\infty |I(x)| \, dx = \int_0^\infty |\mathcal{H}_0 \mathcal{T}_{f'}(x)| \, dx + O(\|f'\|_{H_0^1(\mathbb{R}_+)}).$$

On the other hand, it is proved in [7] that

$$\int_0^\infty |I(x)| \, dx = O(\|f'\|_{H_0^1(\mathbb{R}_+)}).$$

This leads to

Proposition 1. If g is an integrable odd function, then

$$\|\mathcal{H}_0 \mathcal{T}_g\|_{L^1(\mathbb{R}_+)} \lesssim \|g\|_{H_0^1(\mathbb{R}_+)}.$$

The above proof of Proposition 1 looks "artificial". A direct proof, preferable simple enough will be very desirable. In any case, this implies an updated chain of embeddings

$$(24) \quad H_0^1(\mathbb{R}_+) \subseteq H_Q^1(\mathbb{R}_+) \subseteq Q_0 \subseteq L_0^1(\mathbb{R}_+).$$

It is very interesting to figure out which of these embeddings are proper. Correspondingly, intermediate spaces are of interest, both theoretical and practical.

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